

DILATIONS OF FRAMES, OPERATOR VALUED MEASURES AND BOUNDED LINEAR MAPS

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ABSTRACT. We will give an outline of the main results in our recent AMS Memoir, and include some new results, exposition and open problems. In that memoir we developed a general dilation theory for operator valued measures acting on Banach spaces where operator-valued measures (or maps) are not necessarily completely bounded. The main results state that any operator-valued measure, not necessarily completely bounded, always has a dilation to a projection-valued measure acting on a Banach space, and every bounded linear map, again not necessarily completely bounded, on a Banach algebra has a bounded homomorphism dilation acting on a Banach space. Here the dilation space often needs to be a Banach space even if the underlying space is a Hilbert space, and the projections are idempotents that are not necessarily self-adjoint. These results lead to some new connections between frame theory and operator algebras, and some of them can be considered as part of the investigation about “noncommutative” frame theory.

1. INTRODUCTION

Frame theory belongs to the area of applied harmonic analysis, but its underpinnings involve large areas of functional analysis including operator theory, and deep connections with the theory of operator algebras on Hilbert space. For instance, recently the Kadison-Singer “Extension of pure states on von Neumann algebras” problem has been solved and its solution is known to have wide ramifications in frame theory due mainly to the research and excellent exposition of Casazza and of Weaver. The purpose of the present article is to give a good exposition of some recent work of the authors that establishes some deep connections between frame theory on the one hand and operator-valued measures and maps between von Neumann algebras on the other hand. While our work has little or nothing to do directly with the above-mentioned extension of pure states problem, it represents a separate instance of a rather deep connection between frame theory and operator algebras, and this is the point of this article. We show that it may have something to do with another problem of Kadison: the “similarity problem”. At the least it indicates the possibility of another possible approach to that problem. And it does show that the ideas implicit in frame theory belong to the underpinnings of a significant part of modern mathematics.

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Before discussing our results from [18] we need an exposition of some of the preliminaries leading up to them.

2. FRAMES, FRAMINGS, AND OPERATOR VALUED MEASURES

A *frame* \mathcal{F} for a Hilbert space \mathcal{H} is a sequence of vectors $\{x_n\} \subset \mathcal{H}$ indexed by a countable index set \mathbb{J} for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,

$$(2.1) \quad A\|x\|^2 \leq \sum_{n \in \mathbb{J}} |\langle x, x_n \rangle|^2 \leq B\|x\|^2$$

The optimal constants are known as the upper and lower *frame bounds*. A frame is called *tight* if $A = B$, and is called a *Parseval frame* if $A = B = 1$. If we only require that a sequence $\{x_n\}$ satisfies the upper bound condition in (2.1), then $\{x_n\}$ is called a *Bessel sequence*. A frame which is a basis is called a Riesz basis. Orthonormal bases are special cases of Parseval frames. A Parseval frame $\{x_n\}$ for a Hilbert space \mathcal{H} is an orthonormal basis if and only if each x_n is a unit vector.

For a Bessel sequence $\{x_n\}$, its *analysis operator* Θ is a bounded linear operator from \mathcal{H} to $\ell^2(\mathbb{N})$ defined by

$$(2.2) \quad \Theta x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle e_n,$$

where $\{e_n\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$. It is easily verified that

$$\Theta^* e_n = x_n, \quad \forall n \in \mathbb{N}$$

The Hilbert space adjoint Θ^* is called the *synthesis operator* for $\{x_n\}$. The positive operator $S := \Theta^* \Theta : \mathcal{H} \rightarrow \mathcal{H}$ is called the *frame operator*, or sometimes the *Bessel operator* if the Bessel sequence is not a frame, and we have

$$(2.3) \quad Sx = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle x_n, \quad \forall x \in \mathcal{H}.$$

A sequence $\{x_n\}$ is a frame for \mathcal{H} if and only if its analysis operator Θ is bounded, injective and has closed range, which is, in turn, equivalent to the condition that the frame operator S is bounded and invertible. In particular, $\{x_n\}$ is a Parseval frame for \mathcal{H} if and only if Θ is an isometry or equivalently if $S = I$.

Let S be the frame operator for a frame $\{x_n\}$. Then the lower frame bound is $1/\|S^{-1}\|$ and the upper frame bound is $\|S\|$. From (2.3) we obtain the *reconstruction formula* (or *frame decomposition*):

$$x = \sum_{n \in \mathbb{N}} \langle x, S^{-1} x_n \rangle x_n, \quad \forall x \in \mathcal{H}$$

or equivalently

$$x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle S^{-1} x_n, \quad \forall x \in \mathcal{H}.$$

The frame $\{S^{-1} x_n\}$ is called the *canonical or standard dual* of $\{x_n\}$. In the case that $\{x_n\}$ is a Parseval frame for \mathcal{H} , we have that $S = I$ and hence $x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle x_n$, $\forall x \in \mathcal{H}$. More generally, if a Bessel sequence $\{y_n\}$ satisfies $x = \sum_{n \in \mathbb{N}} \langle x, y_n \rangle x_n$, $\forall x \in \mathcal{H}$, where

the convergence is in norm of \mathcal{H} , then $\{y_n\}$ is called an *alternate dual* of $\{x_n\}$. (Then $\{y_n\}$ is also necessarily a frame.) The canonical and alternate duals are often simply referred to as *duals*, and $\{x_n, y_n\}$ is called a *dual frame pair*. It is a well-known fact that a frame $\{x_n\}$ is a Riesz basis if and only if $\{x_n\}$ has a unique dual frame.

There is a geometric interpretation of Parseval frames and general frames. Let P be an orthogonal projection from a Hilbert space \mathcal{K} onto a closed subspace \mathcal{H} , and let $\{u_n\}$ be a sequence in \mathcal{K} . Then $\{Pu_n\}$ is called the *orthogonal compression* of $\{u_n\}$ under P , and correspondingly $\{u_n\}$ is called an *orthogonal dilation* of $\{Pu_n\}$. We first observe that if $\{u_n\}$ is a frame for \mathcal{K} , then $\{Pu_n\}$ is a frame for \mathcal{H} with frame bounds at least as good as those of $\{u_n\}$ (in the sense that the lower frame cannot decrease and the upper bound cannot increase). In particular, $\{Pu_n\}$ is a Parseval frame for \mathcal{H} when $\{u_n\}$ is an orthonormal basis for \mathcal{K} , i.e., every orthogonal compression of an orthonormal basis (resp. Riesz basis) is a Parseval frame (resp. frame) for the projection subspace. The converse is also true: every frame can be orthogonally dilated to a Riesz basis, and every Parseval frame can be dilated to an orthonormal basis. This was apparently first shown explicitly by Han and Larson in Chapter 1 of [17]. There, with appropriate definitions it had an elementary two-line proof. And as noted by several authors, it can be alternately derived by applying the Naimark (Neumark) Dilation theorem for operator valued measures by first passing from a frame sequence to a natural discrete positive operator-valued measure on the power set of the index set. So it is sometimes referred to as the Naimark dilation theorem for frames. In fact, this is the observation that inspired much of the work in [18].

For completeness we formally state this result:

Proposition 2.1. [17] *Let $\{x_n\}$ be a sequence in a Hilbert space \mathcal{H} . Then*

- (1) *$\{x_n\}$ is a Parseval frame for \mathcal{H} if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{u_n\}$ for \mathcal{K} such that $x_n = Pu_n$, where P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*
- (2) *$\{x_n\}$ is a frame for \mathcal{H} if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{v_n\}$ for \mathcal{K} such that $x_n = Pv_n$, where P again is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*

The above dilation result was later generalized in [4] to dual frame pairs.

Theorem 2.2. *Suppose that $\{x_n\}$ and $\{y_n\}$ are two frames for a Hilbert space \mathcal{H} . Then the following are equivalent:*

- (1) *$\{y_n\}$ is a dual for $\{x_n\}$;*
- (2) *There exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_n\}$ for \mathcal{K} such that $x_n = Pu_n$, and $y_n = Pu_n^*$, where $\{u_n^*\}$ is the (unique) dual of the Riesz basis $\{u_n\}$ and P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*

As in [4], a *framing* for a Banach space X is a pair of sequences $\{x_i, y_i\}$ with $\{x_i\}$ in X , $\{y_i\}$ in the dual space X^* of X , satisfying the condition that

$$x = \sum_i \langle x, y_i \rangle x_i,$$

where this series converges unconditionally for all $x \in X$.

The definition of a framing is a natural generalization of the definition of a dual frame pair. Assume that $\{x_i\}$ is a frame for \mathcal{H} and $\{y_i\}$ is a dual frame for $\{x_i\}$. Then $\{x_i, y_i\}$ is clearly a framing for \mathcal{H} . Moreover, if α_i is a sequence of non-zero constants, then $\{\alpha_i x_i, \bar{\alpha}_i^{-1} y_i\}$ (called a rescaling of the pair) is also a framing, although it is easy to show that it need not be a pair of frames, even if $\{\alpha_i x_i\}$, $\{\bar{\alpha}_i^{-1} y_i\}$ are bounded sequence.

We recall that a sequence $\{z_i\}$ in a Banach space Z is called a *Schauder basis* (or just a *basis*) for Z if for each $z \in Z$ there is a unique sequence of scalars $\{\alpha_i\}$ so that $z = \sum_i \alpha_i z_i$. The unique elements $z_i^* \in Z^*$ satisfying

$$(2.4) \quad z = \sum_i z_i^*(z) z_i,$$

for all $z \in Z$, are called the *dual* (or *biorthogonal*) *functionals* for $\{z_i\}$. If the series in (2.4) converges unconditionally for every $z \in Z$, we call $\{z_i, z_i^*\}$ an *unconditional basis* for Z .

We also have an *unconditional basis constant* for an unconditional basis given by:

$$\text{UBC}(z_i) = \sup\{\|\sum_i b_i z_i\| : \|\sum_i a_i z_i\| = 1, |b_i| \leq |a_i|, \forall i\}.$$

If $\{z_i, z_i^*\}$ is an unconditional basis for Z , we can define an equivalent norm on Z by:

$$\|\sum_i a_i z_i\|_1 = \sup\{\|\sum_i b_i a_i z_i\| : |b_i| \leq 1, \forall i\}.$$

Then $\{z_i, z_i^*\}$ is an unconditional basis for Z with $\text{UBC}(z_i) = 1$. In this case we just call $\{z_i\}$ a 1-unconditional basis for Z .

Definition 2.3. [4] A sequence $\{x_i\}_{i \in \mathbb{N}}$ in a Banach space X is a *projective frame* for X if there is a Banach space Z with an unconditional basis $\{z_i, z_i^*\}$ with $X \subset Z$ and a (onto) projection $P : Z \rightarrow X$ so that $Pz_i = x_i$ for all $i \in \mathbb{N}$. If $\{z_i\}$ is a 1-unconditional basis for Z and $\|P\| = 1$, we will call $\{x_i\}$ a *projective Parseval* frame for X .

In this case, we have for all $x \in X$ that

$$x = \sum_i \langle x, z_i^* \rangle z_i = Px = \sum_i \langle x, z_i^* \rangle Pz_i = \sum_i \langle x, z_i^* \rangle x_i,$$

and this series converges unconditionally in X . So this definition recaptures the unconditional convergence from the Hilbert space definition.

We note that there exist projective frames in the sense of Definition 2.3 for an infinite dimensional Hilbert space that fail to be frames. We think they occur in abundance, but specific examples are hard to prove. A concrete example is contained in [18, Chapter 5].

Definition 2.4. [4] A framing model is a Banach space Z with a fixed unconditional basis $\{e_i\}$ for Z . A framing modeled on $(Z, \{e_i\}_{i \in \mathbb{N}})$ for a Banach space X is a pair of sequences $\{y_i\}$ in X^* and $\{x_i\}$ in X so that the operator $\theta : X \rightarrow Z$ defined by

$$\theta u = \sum_{i \in \mathbb{N}} \langle u, y_i \rangle e_i,$$

is an into isomorphism and $\Gamma : Z \rightarrow X$ given by

$$\Gamma(\sum_{i \in \mathbb{N}} a_i e_i) = \sum_{i \in \mathbb{N}} a_i x_i$$

is bounded and $\Gamma\theta = I_X$.

In this setting, Γ becomes the reconstruction operator for the frame. The following result due to Casazza, Han and Larson [4] shows that these three methods for defining a frame on a Banach space are really the same.

Proposition 2.5. *Let X be a Banach space and $\{x_i\}$ be a sequence of elements of X . The following are equivalent:*

- (1) $\{x_i\}$ is a projective frame for X .
- (2) There exists a sequence $y_i \in X^*$ so that $\{x_i, y_i\}$ is a framing for X .
- (3) There exists a sequence $y_i \in X^*$ and a framing model $(Z, \{e_i\})$ so that $\{x_i, y_i\}$ is a framing modeled on $(Z, \{e_i\})$.

The proof of the implication from (1) to (2) is trivial: If $\{z_i\}$ is an unconditional basis for a Banach space Z and P is a bounded projection from Z to a closed subspace X with $x_i = Pe_i$, then (x_i, y_i) is a framing for X , where $y_i = P^*z_i^*$ and $\{z_i^*\}$ is the (unique) dual basis of $\{z_i\}$. One of the main contributions of paper [4] was to show that every framing can be obtained in this way.

Theorem 2.6 (Corollary 4.7 of [4]). *Suppose that $\{x_i, y_i\}$ is a framing for X . Then there exist a Banach space Z containing X and an unconditional basis $\{z_i, z_i^*\}$ for Z such that $x_i = Pz_i$ and $y_i = P^*z_i^*$, where P is a bounded projection from Z onto X .*

The definition of (discrete) frames has a natural generalization.

Definition 2.7. Let \mathcal{H} be a separable Hilbert space and Ω be a σ -locally compact (σ -compact and locally compact) Hausdorff space endowed with a positive Radon measure μ with $\text{supp}(\mu) = \Omega$. A weakly continuous function $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is called a *continuous frame* if there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1\|x\|^2 \leq \int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \leq C_2\|x\|^2, \quad \forall x \in \mathcal{H}.$$

If $C_1 = C_2$ then the frame is called *tight*. Associated to \mathcal{F} is the frame operator $S_{\mathcal{F}} : \mathcal{H} \rightarrow \mathcal{H}$ defined in the weak sense by

$$\langle S_{\mathcal{F}}(x), y \rangle := \int_{\Omega} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega).$$

It follows from the definition that $S_{\mathcal{F}}$ is a bounded, positive, and invertible operator. We define the following transform associated to \mathcal{F} ,

$$V_{\mathcal{F}} : \mathcal{H} \rightarrow L^2(\Omega, \mu), \quad V_{\mathcal{F}}(x)(\omega) := \langle x, \mathcal{F}(\omega) \rangle.$$

This operator is called the *analysis operator* in the literature and its adjoint operator is given by

$$V_{\mathcal{F}}^* : L^2(\Omega, \mu) \rightarrow \mathcal{H}, \quad \langle V_{\mathcal{F}}^*(f), x \rangle := \int_{\Omega} f(\omega) \langle \mathcal{F}(\omega), x \rangle d\mu(\omega).$$

Then we have $S_{\mathcal{F}} = V_{\mathcal{F}}^* V_{\mathcal{F}}$, and

$$(2.5) \quad \langle x, y \rangle = \int_{\Omega} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{G}(\omega), y \rangle d\mu(\omega),$$

where $\mathcal{G}(\omega) := S_{\mathcal{F}}^{-1}\mathcal{F}(\omega)$ is the *standard dual* of \mathcal{F} . A weakly continuous function $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is called *Bessel* if there exists a positive constant C such that

$$\int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \leq C\|x\|^2, \quad \forall x \in \mathcal{H}.$$

It can be easily shown that if $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is Bessel, then it is a frame for \mathcal{H} if and only if there exists a Bessel mapping \mathcal{G} such that the reconstruction formula (2.5) holds. This \mathcal{G} may not be the standard dual of \mathcal{F} . We will call $(\mathcal{F}, \mathcal{G})$ a *dual pair*.

A discrete frame is a Riesz basis if and only if its analysis operator is surjective. But for a continuous frame \mathcal{F} , in general we don't have the dilation space to be $L^2(\Omega, \mu)$. In fact, this could happen only when μ is purely atomic. Therefore there is no Riesz basis type dilation theory for continuous frames (however, we will see later that in contrast the induced operator-valued measure does have a projection valued measure dilation). The following modified dilation theorem is due to Gabardo and Han [12]:

Theorem 2.8. *Let \mathcal{F} be a (Ω, μ) -frame for \mathcal{H} and \mathcal{G} be one of its duals. Suppose that both $V_{\mathcal{F}}(\mathcal{H})$ and $V_{\mathcal{G}}(\mathcal{H})$ are contained in the range space \mathcal{M} of the analysis operator for some (Ω, μ) -frame. Then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a (Ω, μ) -frame $\tilde{\mathcal{F}}$ for \mathcal{K} with $P\tilde{\mathcal{F}} = \mathcal{F}$, $P\tilde{\mathcal{G}} = \mathcal{G}$ and $V_{\tilde{\mathcal{F}}}(\mathcal{H}) = \mathcal{M}$, where $\tilde{\mathcal{G}}$ is the standard dual of $\tilde{\mathcal{F}}$ and P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*

Let Ω be a compact Hausdorff space, and let \mathcal{B} be the σ -algebra of all the Borel subsets of Ω . A $B(\mathcal{H})$ -valued measure on Ω is a mapping $E : \mathcal{B} \rightarrow B(\mathcal{H})$ that is weakly countably additive, i.e., if $\{B_i\}$ is a countable collection of disjoint Borel sets with union B , then

$$\langle E(B)x, y \rangle = \sum_i \langle E(B_i)x, y \rangle$$

holds for all x, y in \mathcal{H} . The measure is called *bounded* provided that

$$\sup\{\|E(B)\| : B \in \mathcal{B}\} < \infty,$$

and we let $\|E\|$ denote this supremum. The measure is called *regular* if for all x, y in \mathcal{H} , the complex measure given by

$$(2.6) \quad \mu_{x,y}(B) = \langle E(B)x, y \rangle$$

is regular.

Given a regular bounded $B(\mathcal{H})$ -valued measure E , one obtains a bounded, linear map

$$\phi_E : C(\Omega) \rightarrow B(\mathcal{H})$$

by

$$(2.7) \quad \langle \phi_E(f)x, y \rangle = \int_{\Omega} f d\mu_{x,y}.$$

Conversely, given a bounded, linear map $\phi : C(\Omega) \rightarrow B(\mathcal{H})$, if one defines regular Borel measures $\{\mu_{x,y}\}$ for each x, y in \mathcal{H} by the above formula (2.7), then for each Borel set B , there exists a unique, bounded operator $E(B)$, defined by formula (2.6), and the map $B \rightarrow E(B)$ defines a bounded, regular $B(\mathcal{H})$ -valued measure. There is a one-to-one correspondence between the bounded, linear maps of $C(\Omega)$ into $B(\mathcal{H})$ and the regular bounded $B(\mathcal{H})$ -valued measures. Such measures are called

- (i) *spectral* if $E(B_1 \cap B_2) = E(B_1) \cdot E(B_2)$,
- (ii) *positive* if $E(B) \geq 0$,
- (iii) *self-adjoint* if $E(B)^* = E(B)$,

for all Borel sets B, B_1 and B_2 . Note that if E is spectral and self-adjoint, then $E(B)$ must be an orthogonal projection for all $B \in \mathcal{B}$, and hence E is positive.

In the commutative C^* theory, compactness is usually used as above because when viewing a unital C^* -algebra as $C(\Omega)$ there is no loss in generality in taking Ω to be compact, because if needed it can be taken to be $\beta\Omega$ – the Stone-Cech compactification of Ω . This is because the C^* -algebras $C(\Omega)$ and $C(\beta\Omega)$ are $*$ -isomorphic. Having Ω compact makes the integration theory representation of linear maps and the connection between linear maps on $C(\Omega)$ and operator valued measures very elegant.

But in our theory, the basic connection to frame theory is essentially lost if we replace the index set of the frame with its Stone-cech compactification. In the continous frame case it is more natural to assume Ω is σ -locally compact (as in Definition 2.7), and in the general dilation theory we need to use the *general* measurable space setting (as in Definition 3.2) to preserve our basic connections with the frame theory.

Both discrete and continuous framings induce operator valued measures in a natural way.

Example 2.9. Let $\{x_i\}_{i \in \mathbb{J}}$ be a frame for a separable Hilbert space \mathcal{H} . Let Σ be the σ -algebra of all subsets of \mathbb{J} . Define the mapping

$$E : \Sigma \rightarrow B(\mathcal{H}), \quad E(B) = \sum_{i \in B} x_i \otimes x_i$$

where $x \otimes y$ is the mapping on \mathcal{H} defined by $(x \otimes y)(u) = \langle u, y \rangle x$. Then E is a regular, positive $B(\mathcal{H})$ -valued measure.

Similarly, suppose that $\{x_i, y_i\}_{i \in \mathbb{J}}$ is a non-zero framing for a separable Hilbert space \mathcal{H} . Define the mapping

$$E : \Sigma \rightarrow B(\mathcal{H}), \quad E(B) = \sum_{i \in B} x_i \otimes y_i,$$

for all $B \in \Sigma$. Then E is a $B(\mathcal{H})$ -valued measure.

Example 2.10. Let X be a Banach space and Ω be a σ -locally compact Hausdorff space. Let μ be a Borel measure on Ω . A continuous framing on X is a pair of maps $(\mathcal{F}, \mathcal{G})$,

$$\mathcal{F} : \Omega \rightarrow X, \quad \mathcal{G} : \Omega \rightarrow X^*,$$

such that the equation

$$\langle E_{(\mathcal{F}, \mathcal{G})}(B)x, y \rangle = \int_B \langle x, \mathcal{G}(\omega) \rangle \langle \mathcal{F}(\omega), y \rangle d\mu(\omega)$$

for $x \in X$, $y \in X^*$, and B a Borel subset of Ω , defines an operator-valued probability measure on Ω taking value in $B(X)$. In particular, we require the integral on the right to converge for each $B \subset \Omega$. We have

$$(2.8) \quad E_{(\mathcal{F}, \mathcal{G})}(B) = \int_B \mathcal{F}(\omega) \otimes \mathcal{G}(\omega) dE(\omega)$$

where the integral converges in the sense of Bochner. In particular, since $E_{(\mathcal{F}, \mathcal{G})}(\Omega) = I_X$, we have for any $x \in X$ that

$$\langle x, y \rangle = \int_{\Omega} \langle x, \mathcal{G}(\omega) \rangle \langle \mathcal{F}(\omega), y \rangle dE(\omega).$$

Remark 2.11. We point out that there exists an operator space with a (finite dimensional) projection valued (purely atomic) probability measure that does not admit a framing. Let X be the space of all compact operators T on ℓ_2 which have a triangular representing matrix with respect to the unit vector basis, i.e.

$$Te_n = \sum_{m=1}^n a_{n,m} e_m$$

for all $n \in \mathbb{N}$. Let X_n be the subspace of X consisting of those $T \in X$ such that $Te_j = 0$ for $j \neq n$ (i.e. for which $a_{j,m} = 0$ unless $j = n$). It is clear that X_n is isometric to ℓ_2^n , $n = 1, 2, \dots$. Moreover, it is trivial to check that $\{X_n\}_{n=1}^{\infty}$ forms an unconditional finite dimensional decomposition of X which naturally induces a projection valued probability measure. Let P_n be the canonical projection from X onto X_n satisfy:

- (i) $\dim(P_n(X)) = \dim(X_n) = n$ for all $n \in \mathbb{N}$;
- (ii) $P_n P_m = P_m P_n = 0$ for any $n \neq m \in \mathbb{N}$;
- (iii) $x = \sum_{n=1}^{\infty} P_n(x)$ for every $x \in X$.

Let Σ be the σ -algebra of all subsets of \mathbb{N} . Define $E : \Sigma \rightarrow B(X)$ by $E(B) = \sum_{n \in B} P_n$. Then E is a projection valued probability measure with $\dim(E(\{n\})) = n$. Nevertheless, it follows from the results of [13] that X does not have an unconditional basis and it is not even complemented in a space with an unconditional basis. Thus, by Proposition 2.5, X does not have a framing.

Let \mathcal{A} be a unital C^* -algebra. An operator-valued linear map $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ is said to be *positive* if $\phi(a^*a) \geq 0$ for every $a \in \mathcal{A}$, and it is called *completely positive* (cp for abbreviation) if for every n -tuple a_1, \dots, a_n of elements in \mathcal{A} , the matrix $(\phi(a_i^* a_j))$ is positive in the usual sense that for every n -tuple of vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have

$$(2.9) \quad \sum_{i,j=1}^n \langle \phi(a_i a_j^*) \xi_j, \xi_i \rangle \geq 0$$

or equivalently, $(\phi(a_i^* a_j))$ is a positive operator on the Hilbert space $\mathcal{H} \otimes \mathbb{C}^n$ ([23]).

Let \mathcal{A} be a C^* -algebra. We use M_n to denote the set of all $n \times n$ complex matrices, and $M_n(\mathcal{A})$ to denote the set of all $n \times n$ matrices with entries from \mathcal{A} . Given two C^* -algebras \mathcal{A} and \mathcal{B} and a map $\phi : \mathcal{A} \rightarrow \mathcal{B}$, obtain maps $\phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ via the formula

$$\phi_n((a_{i,j})) = (\phi(a_{i,j})).$$

The map ϕ is called *completely bounded* (cb for abbreviation) if ϕ is bounded and $\|\phi\|_{cb} = \sup_n \|\phi_n\|$ is finite.

3. DILATIONS OF OPERATOR VALUED MEASURES

Possibly the first well-known dilation result for operator valued measures is due to Naimark.

Theorem 3.1 (Naimark's Dilation Theorem). *Let E be a regular, positive, $B(\mathcal{H})$ -valued measure on Ω . Then there exist a Hilbert space \mathcal{K} , a bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$, and a regular, self-adjoint, spectral, $B(\mathcal{K})$ -valued measure F on Ω , such that*

$$E(B) = V^* F(B) V.$$

From Naimark's dilation Theorem, we know that every regular positive operator-valued measure (OVM for abbreviation) can be dilated to a self-adjoint, spectral operator-valued measure on a larger Hilbert space. But not all of the operator-valued measures can have a Hilbert dilation space. Such an example was constructed in [18] in which we constructed an operator-valued measure induced by the framing that does not have a Hilbert dilation space. The construction is based on an example of Osaka [22] of a normal non-completely bounded map from $\ell^\infty(\mathbb{N})$ into $B(H)$. In fact operator valued measures that admit Hilbert space dilations are the ones that are closely related to completely bounded measures and maps.

Now let Ω be a compact Hausdorff space, let E be a bounded, regular, operator-valued measure on Ω , and let $\phi : C(\Omega) \rightarrow B(\mathcal{H})$ be the bounded, linear map associated with E by integration. So for any $f \in C(\Omega)$,

$$\langle \phi(f)x, y \rangle = \int_{\Omega} f d\mu_{x,y},$$

where

$$\mu_{x,y}(B) = \langle E(B)x, y \rangle$$

The OVM E is called completely bounded when ϕ is completely bounded. Using Wittstock's decomposition theorem, E is completely bounded if and only if it can be expressed as a linear combination of positive operator-valued measures.

Let $\{x_i\}_{i \in \mathbb{J}}$ be a non-zero frame for a separable Hilbert space \mathcal{H} . Let Σ be the σ -algebra of all subsets of \mathbb{J} , and

$$E : \Sigma \rightarrow B(\mathcal{H}), \quad E(B) = \sum_{i \in B} x_i \otimes x_i$$

Since E is a regular, positive $B(\mathcal{H})$ -valued measure, by Naimark's dilation Theorem 3.1, there exists a Hilbert space \mathcal{K} , a bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$, and a regular, self-adjoint, spectral, $B(\mathcal{K})$ -valued measure F on \mathbb{J} , such that

$$E(B) = V^* F(B) V.$$

This Hilbert space \mathcal{K} can be ℓ_2 , and the atoms $x_i \otimes x_i$ of the measure dilate to rank-1 projections $e_i \otimes e_i$, where $\{e_i\}$ is the standard orthonormal basis for ℓ^2 . That is \mathcal{K} can be the same as the dilation space in Proposition 2.1 (ii).

In the case that $\{x_i, y_i\}_{i \in \mathbb{J}}$ is a non-zero framing for a separable Hilbert space \mathcal{H} , and $E(B) = \sum_{i \in B} x_i \otimes y_i$ for all $B \in \Sigma$, E is a $B(\mathcal{H})$ -valued measure. In [18] we showed that this E also has a dilation space Z . But this dilation space is not necessarily a Hilbert space, in general, it is a Banach space and consistent with Proposition 2.5. The dilation is essentially constructed using Proposition 2.5 (ii), where the dilation of the atoms $x_i \otimes y_i$ corresponds to the projection $u_i \otimes u_i^*$ and $\{u_i\}$ is an unconditional basis for the dilation space Z .

Framings are the natural generalization of discrete frame theory (more specifically, dual-frame pairs) to non-Hilbertian settings. Even if the underlying space is a Hilbert space, the dilation space for framing induced operator valued measures can fail to be Hilbertian. This theory was originally developed by Casazza, Han and Larson in [4] as an attempt to introduce *frame theory with dilations* into a Banach space context. The initial motivation of this investigation was to completely understand the dilation theory of framings. In the context of Hilbert spaces, we realized that the dilation theory for discrete framings from [4] induces a dilation theory for discrete operator valued measures that may fail to be completely bounded.

These examples inspired us to consider Banach space dilation theory for arbitrary operator valued measures.

Definition 3.2. Let X and Y be Banach spaces, and let (Ω, Σ) be a measurable space. A $B(X, Y)$ -valued measure on Ω is a map $E : \Sigma \rightarrow B(X, Y)$ that is countably additive in the weak operator topology; that is, if $\{B_i\}$ is a disjoint countable collection of members of Σ with union B , then

$$y^*(E(B)x) = \sum_i y^*(E(B_i)x)$$

for all $x \in X$ and $y^* \in Y^*$.

We will use the symbol (Ω, Σ, E) if the range space is clear from context, or $(\Omega, \Sigma, E, B(X, Y))$, to denote this operator-valued measure system.

The Orlicz-Pettis theorem states that weak unconditional convergence and norm unconditional convergence of a series are the same in every Banach space (c.f. [8]). Thus we have that $\sum_i E(B_i)x$ weakly unconditionally converges to $E(B)x$ if and only if $\sum_i E(B_i)x$ strongly unconditionally converges to $E(B)x$. So Definition 3.2 is equivalent to saying that E is strongly countably additive, that is, if $\{B_i\}$ is a disjoint countable collection of members of Σ with union B , then

$$E(B)x = \sum_i E(B_i)x, \quad \forall x \in X.$$

Definition 3.3. Let E be a $B(X, Y)$ -valued measure on (Ω, Σ) . Then the *norm of E* is defined by

$$\|E\| = \sup_{B \in \Sigma} \|E(B)\|.$$

We call E *normalized* if $\|E\| = 1$.

A $B(X, Y)$ -valued measure E is always bounded, i.e.

$$(3.1) \quad \sup_{B \in \Sigma} \|E(B)\| < +\infty.$$

Indeed, for all $x \in X$ and $y^* \in Y^*$, $\mu_{x, y^*}(B) := y^*(E(B)x)$ is a complex measure on (Ω, Σ) . From complex measure theory (c.f. [25]), we know that μ_{x, y^*} is bounded, i.e.

$$\sup_{B \in \Sigma} |y^*(E(B)x)| < +\infty.$$

By the Uniform Boundedness Principle, we get (3.1).

Similar to the Hilbert space operator valued measures, we introduce the following definitions.

Definition 3.4. A $B(X)$ -valued measure E on (Ω, Σ) is called:

- (i) an *operator-valued probability measure* if $E(\Omega) = I_X$,
- (ii) a *projection-valued measure* if $E(B)$ is a projection on X for all $B \in \Sigma$,
- (iii) a *spectral operator-valued measure* if for all $A, B \in \Sigma$, $E(A \cap B) = E(A) \cdot E(B)$
(we will also use the term *idempotent-valued measure* to mean a spectral-valued measure.)

For general operator valued measures we established the following dilation theorem [18].

Theorem 3.5. Let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure. Then there exist a Banach space Z , bounded linear operators $S : Z \rightarrow Y$ and $T : X \rightarrow Z$, and a projection-valued probability measure $F : \Sigma \rightarrow B(Z)$ such that

$$E(B) = SF(B)T$$

for all $B \in \Sigma$.

We will call (F, Z, S, T) in the above theorem a *Banach space dilation system*, and a *Hilbert dilation system* if Z can be taken as a Hilbert space. This theorem generalizes Naimark's (Neumark's) Dilation Theorem for positive operator valued measures. But even in the case that the underlying space is a Hilbert space the dilation space cannot always be taken to be a Hilbert space. Thus elements of the theory of Banach spaces are essential in this work.

A key idea is the introduction of the elementary dilation space and the minimal dilation norm.

Let X, Y be Banach spaces and $(\Omega, \Sigma, E, B(X, Y))$ an operator-valued measure system. For any $B \in \Sigma$ and $x \in X$, define

$$E_{B,x} : \Sigma \rightarrow Y, \quad E_{B,x}(A) = E(B \cap A)x, \quad \forall A \in \Sigma.$$

Then it is easy to see that $E_{B,x}$ is a vector-valued measure on (Ω, Σ) of Y and $E_{B,x} \in \mathfrak{M}_\Sigma^Y$.

Let $M_E = \text{span}\{E_{B,x} : x \in X, B \in \Sigma\}$. We introduce some linear mappings on the spaces X, Y and M_E .

For any $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{B_i\}_{i=1}^N \subset \Sigma$ and $\{x_i\}_{i=1}^N \subset X$, the mappings

$$S : M_E \rightarrow Y, \quad S\left(\sum_{i=1}^N C_i E_{B_i, x_i}\right) = \sum_{i=1}^N C_i E(B_i)x_i$$

$$T : X \rightarrow M_E, \quad T(x) = E_{\Omega, x}$$

and

$$F(B) : M_E \rightarrow M_E, \quad F(B)\left(\sum_{i=1}^N C_i E_{B_i, x_i}\right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i}, \quad \forall B \in \Sigma$$

are well-defined and linear.

Definition 3.6. Let M_E be the space induced by $(\Omega, \Sigma, E, B(X, Y))$. Let $\|\cdot\|$ be a norm on M_E . Denote this normed space by $M_{E, \|\cdot\|}$ and its completion $\widetilde{M}_{E, \|\cdot\|}$. The norm on $\widetilde{M}_{E, \|\cdot\|}$, with $\|\cdot\| := \|\cdot\|_{\mathcal{D}}$ given by a norming function \mathcal{D} as discussed above, is called a *dilation norm of E* if the following conditions are satisfied:

(i) The mapping $S_{\mathcal{D}} : \widetilde{M}_{E,\mathcal{D}} \rightarrow Y$ defined on M_E by

$$S_{\mathcal{D}}\left(\sum_{i=1}^N C_i E_{B_i, x_i}\right) = \sum_{i=1}^N C_i E(B_i) x_i$$

is bounded.

(ii) The mapping $T_{\mathcal{D}} : X \rightarrow \widetilde{M}_{E,\mathcal{D}}$ defined by

$$T_{\mathcal{D}}(x) = E_{\Omega, x}$$

is bounded.

(iii) The mapping $F_{\mathcal{D}} : \Sigma \rightarrow B(\widetilde{M}_{E,\mathcal{D}})$ defined by

$$F_{\mathcal{D}}(B)\left(\sum_{i=1}^N C_i E_{B_i, x_i}\right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i}$$

is an operator-valued measure,

where $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$.

We call the Banach space $\widetilde{M}_{E,\mathcal{D}}$ the *elementary dilation space of E* and

$$(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{E,\mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$$

the elementary dilation operator-valued measure system. The *minimal dilation norm* $\|\cdot\|_{\alpha}$ on M_E is defined by

$$\left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\alpha} = \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y$$

for all $\sum_{i=1}^N C_i E_{B_i, x_i} \in M_E$. Using this we show that every OVM has a projection valued dilation to an elementary dilation space, and moreover, $\|\cdot\|_{\alpha}$ is a minimal norm on the elementary dilation space.

A corresponding dilation projection-valued measure system $(\Omega, \Sigma, F, B(Z), S, T)$ is said to be *injective* if $\sum F(B_i)T(x_i) = 0$ whenever $\sum E_{B_i, x_i} = 0$.

It is useful to note that all the elementary dilation spaces are Banach spaces of functions.

Theorem 3.7. *Let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure and (F, Z, S, T) be an injective Banach space dilation system. Then we have the following:*

(i) *There exist an elementary Banach space dilation system $(F_{\mathcal{D}}, \widetilde{M}_{E,\mathcal{D}}, S_{\mathcal{D}}, T_{\mathcal{D}})$ of E and a linear isometric embedding*

$$U : \widetilde{M}_{E,\mathcal{D}} \rightarrow Z$$

such that

$$S_{\mathcal{D}} = SU, \quad F(\Omega)T = UT_{\mathcal{D}}, \quad UF_{\mathcal{D}}(B) = F(B)U, \quad \forall B \in \Sigma.$$

(ii) *The norm $\|\cdot\|_{\alpha}$ is indeed a dilation norm. Moreover, If \mathcal{D} is a dilation norm of E , then there exists a constant $C_{\mathcal{D}}$ such that for any $\sum_{i=1}^N C_i E_{B_i, x_i} \in M_{E,\mathcal{D}}$,*

$$\sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y \leq C_{\mathcal{D}} \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{D}},$$

where $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. Consequently

$$\|f\|_\alpha \leq C_{\mathcal{D}} \|f\|_{\mathcal{D}}, \quad \forall f \in M_E.$$

Definition 3.8. Let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure and (F, Z, S, T) be a Banach space dilation system. Then (F, Z, S, T) is called *linearly minimal* if Z is the closed linear span of $F(\Sigma)TX$, where $F(\Sigma)TX = \{F(B)(Tx) : B \in \Sigma, x \in X\}$.

A projection valued measure can have a nontrivial linearly minimal dilation to another projection valued measure. The following simple example illustrates this. It is also an example of a dilation projection-valued measure system which is not injective and for which the conclusion of Theorem 3.7 is not true. This shows that if we drop the “injectivity” in the hypothesis of Theorem 3.7 the conclusion need not be true. However a simple modification of the conclusion will be true (see Remark 3.10).

Example 3.9. Let (Ω, Σ, μ) be a probability space and let ν be a finite measure that dominates μ . Let $X := L^2(\Omega, \mu)$ and let $Y := L^2(\Omega, \nu)$. Let α be a bounded linear functional on X that takes 1 at the function $\eta = 1$. Let $\Omega = \Omega_0^c \cup \Omega_0$ be the Hahn decomposition, where Ω_0 is a measurable subset of Ω which is a null set for μ and which supports the singular part of ν with respect to μ . Regard $L^2(\Omega, \nu)$ as the direct sum of $L^2(\Omega, \mu)$ and $L^2(\Omega_0, \nu)$. Embed X into Y by $T(f) = f \oplus \alpha(f)\chi_{\Omega_0}$, where χ_{Ω_0} is the constant function 1 in $L^2(\Omega_0, \nu)$. Since α is a linear functional T is a linear map. In particular it maps the constant function 1 in $X := L^2(\Omega, \mu)$ to the constant function 1 in $Y := L^2(\Omega, \nu)$. Define a projection valued measure $\phi : \Sigma \rightarrow B(X)$ by setting $\phi(B) = M_{\chi_B}$, the projection operator of multiplication by the characteristic function of B . Do the same construction to define a projection valued measure $\Phi : \Sigma \rightarrow B(Y)$. Since TX contains the constant function 1 in $L^2(\Omega, \nu)$, the closed linear span of $\Phi(\Sigma)TX$ is Y .

Let S denote the mapping of $Y := L^2(\Omega, \nu)$ onto $X := L^2(\Omega, \mu)$ determined by the function mapping $f \rightarrow f|_{\Omega_0^c}$. Then S has kernel $L^2(\Omega_0, \nu)$.

Then Φ is a dilation of ϕ for the dilation maps T and S , and the dilation is linearly minimal because the closed linear span of $\Phi(\Sigma)TX$ is Y . The dilation is clearly non-injective, and the conclusion of Theorem 3.7 fails for it.

Remark 3.10. We have the following natural generalization of Theorem 3.7: Let all terms be as in the hypotheses of Theorem 3.7 except do not assume that the Banach space dilation system (F, Z, S, T) is injective. First, obtain a reduction if necessary by restricting the range space of F so that the closure of the range of F times the range of T is all of Z . This makes the dilation linearly minimal. Example 3.9 shows that this reduction to linearly minimal is not alone sufficient to generalize Theorem 3.7. Obtain a second reduction by replacing Z with its quotient by the kernel of S . Then the hypotheses of Theorem 3.7 are satisfied, so we can obtain a generalization of Theorem 3.7 by removing the injectivity requirement in the hypothesis and inserting the restriction reduction followed by the quotient reduction in the statement of the conclusion. In Example 3.9 the restriction reduction is unnecessary because the dilation is already linearly minimal, and the quotient reduction makes the reduced dilation equivalent to ϕ .

The point of this is that the *minimal elementary norm dilation* of this section is really a *geometrically minimal dilation* in the sense that any dilation, after a simple restriction

reduction and a quotient reduction if necessary, is isometrically isomorphic to an *elementary* dilation norm dilation. And the class of elementary dilation norm spaces are related in the sense that there is a minimal dilation norm and a maximal dilation norm, and all dilation norms lie between the minimal and the maximal norm on the elementary function space, and the actual dilation space is the completion of the elementary function space in one of the dilation norms. So in this sense the minimal norm elementary dilation of an operator valued measure is subordinate to all other dilations of the OVM.

While in general an operator-valued probability measure does not admit a Hilbert space dilation, the dilation theory can be strengthened in the case that it does admit a Hilbert space dilation:

Theorem 3.11. *Let $E : \Sigma \rightarrow B(\mathcal{H})$ be an operator-valued probability measure. If E has a Hilbert dilation system $(\tilde{E}, \tilde{H}, S, T)$, then there exists a corresponding Hilbert dilation system (F, \mathcal{K}, V^*, V) such that $V : \mathcal{H} \rightarrow \mathcal{K}$ is an isometric embedding.*

This theorem turns out to have some interesting applications to framing induced operator valued measure dilation. In particular, it led to a complete characterization of framings whose induced operator valued measures are completely bounded. We include here a few sample examples with the following theorem:

Theorem 3.12. *Let $(x_i, y_i)_{i \in \mathbb{N}}$ be a non-zero framing for a Hilbert space \mathcal{H} , and E be the operator-valued probability measure induced by $(x_i, y_i)_{i \in \mathbb{N}}$. Then we have the following:*

(i) *E has a Hilbert dilation space \mathcal{K} if and only if there exist $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are the frames for the Hilbert space \mathcal{H} .*

(ii) *E is a completely bounded map if and only if $\{x_i, y_i\}_{i \in \mathbb{N}}$ can be re-scaled to dual frames.*

(iii) *If $\inf \|x_i\| \cdot \|y_i\| > 0$, then we can find $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are frames for the Hilbert space \mathcal{H} . Hence the operator-valued measure induced by $\{x_i, y_i\}_{i \in \mathbb{N}}$ has a Hilbertian dilation.*

For the existence of non-rescalable (to dual frame pairs) framings, we obtained the following:

Theorem 3.13. *There exists a framing for a Hilbert space such that its induced operator-valued measure is not completely bounded, and consequently it can not be re-scaled to obtain a framing that admits a Hilbert space dilation.*

The second part of this theorem follows from the first part of Theorem 3.12 (ii).

Remark 3.14. For the existence of such an example, the motivating example of framing constructed by Casazza, Han and Larson (Example 3.9 in [4]) can not be dilated to an unconditional basis for a Hilbert space, although it can be dilated to an unconditional basis for a Banach space. We originally conjectured that this is an example that fails to induce a completely bounded operator valued measure. However, it turns out that this framing can be re-scaled to a framing that admits a Hilbert space dilation, and consequently disproves our conjecture. Our construction of the new example in Theorem 3.13 uses a non-completely bounded map to construct a non-completely bounded OVM which yields the required framing. This delimiting example shows that the dilation theory for framings developed in [4] gives a true generalization of Naimark's Dilation Theorem

for the discrete case. This is the example that led us to consider general (non-necessarily-discrete) operator valued measures, and to the results of Chapter 2 that lead to the dilation theory for general (not necessarily completely bounded) OVM's that completely generalizes Naimark's Dilation theorem in a Banach space setting, and which is new even for Hilbert spaces.

Part (iii) of Theorem 3.12 provides us a sufficient condition under which a framing induced operator-valued measure has a Hilbert space dilation. This can be applied to framings that have nice structures. For example, the following is an unexpected result for unitary system induced framings, where a unitary system is a countable collection of unitary operators. This clearly applies to wavelet and Gabor systems.

Corollary 3.15. *Let \mathcal{U}_1 and \mathcal{U}_2 be unitary systems on a separable Hilbert space \mathcal{H} . If there exist $x, y \in \mathcal{H}$ such that $\{\mathcal{U}_1 x, \mathcal{U}_2 y\}$ is a framing of \mathcal{H} , then $\{\mathcal{U}_1 x\}$ and $\{\mathcal{U}_2 y\}$ both are frames for \mathcal{H} .*

There exist examples of sequences $\{x_n\}$ and $\{y_n\}$ in a Hilbert space \mathcal{H} with the following properties:

- (i) $x = \sum_n \langle x, x_n \rangle y_n$ hold for all x in a dense subset of \mathcal{H} , and the convergence is unconditionally.
- (ii) $\inf \|x_n\| \cdot \|y_n\| > 0$.
- (iii) $\{x_n, y_n\}$ is not a framing.

Example 3.16. Let $\mathcal{H} = L^2[0, 1]$, and $g(t) = t^{1/4}$, $f(t) = 1/g(t)$. Define $x_n(t) = e^{2\pi i n t} f(t)$ and $y_n(t) = e^{2\pi i n t} g(t)$. Then it is easy to verify (i) and (ii). For (iii), we consider the convergence of the series

$$\sum_{n \in \mathbb{Z}} \langle f, x_n \rangle y_n.$$

Note that $\|\langle f, x_n \rangle y_n\|^2 = |\langle f, x_n \rangle|^2 \cdot \|g\|^2$ and $\{\langle f, x_n \rangle\}$ is not in ℓ^2 (since $f^2 \notin L^2[0, 1]$). Thus $\sum_n \langle f, x_n \rangle y_n$ can not be convergent unconditionally. Therefore $\{x_n, y_n\}$ is not a framing.

4. DILATIONS OF BOUNDED LINEAR MAPS

Inspired by the techniques used to build the dilation theory for general operator valued measures we consider establishing a dilation theory for general linear maps. Historically the dilation theory has been extensively investigated in the context of positive, or completely bounded maps on C^* -algebras, with Stinespring's dilation theorem as possibly one of the most notable results in this direction (c.f. [1, 23] and the references therein).

Theorem 4.1. *[Stinespring's dilation theorem] Let \mathcal{A} be a unital C^* -algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space \mathcal{K} , a unital $*$ -homomorphism $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \rightarrow \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^* \pi(a) V.$$

The following is also well known for commutative C^* -algebras:

Theorem 4.2 (cf. Theorem 3.11, [23]). *Let \mathcal{B} be a C^* -algebra, and let $\phi : C(\Omega) \rightarrow \mathcal{B}$ be positive. Then ϕ is completely positive.*

This result together with Theorem 4.1 implies that Stinespring's dilation theorem holds for positive maps when \mathcal{A} is a unital commutative C^* -algebra.

A proof of Naimark's dilation theorem by using Stinespring's dilation theorem can be sketched as follows: Let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be the natural extension of E to the C^* -algebra \mathcal{A} generated by all the characteristic functions of measurable subsets of Ω . Then ϕ is positive, and hence is completely positive by Theorem 4.2. Apply Stinespring's dilation theorem to obtain a $*$ -homomorphism $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, and a bounded, linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$ such that $\phi(f) = V^* \pi(f) V$ for all f in \mathcal{A} . Let F be the $B(\mathcal{K})$ -valued measure corresponding to π . Then it can be verified that F has the desired properties.

Completely positive maps are completely bounded. In the other direction we have Wittstock's decomposition theorem [23]:

Proposition 4.3. *Let \mathcal{A} be a unital C^* -algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely bounded map. Then ϕ is a linear combination of two completely positive maps.*

The following is a generalization of Stinespring's representation theorem.

Theorem 4.4. *Let \mathcal{A} be a unital C^* -algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely bounded map. Then there exists a Hilbert space \mathcal{K} , a $*$ -homomorphism $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \rightarrow \mathcal{K}, i = 1, 2$, with $\|\phi\|_{cb} = \|V_1\| \cdot \|V_2\|$ such that*

$$\phi(a) = V_1^* \pi(a) V_2$$

for all $a \in \mathcal{A}$. Moreover, if $\|\phi\|_{cb} = 1$, then V_1 and V_2 may be taken to be isometries.

Now let Ω be a compact Hausdorff space, let E be a bounded, regular, operator-valued measure on Ω , and let $\phi : C(\Omega) \rightarrow B(\mathcal{H})$ be the bounded, linear map associated with E by integration as described in section 1.4.1. So for any $f \in C(\Omega)$,

$$\langle \phi(f)x, y \rangle = \int_{\Omega} f d\mu_{x,y},$$

where

$$\mu_{x,y}(B) = \langle E(B)x, y \rangle.$$

The OVM E is called completely bounded when ϕ is completely bounded. Using Wittstock's decomposition theorem, E is completely bounded if and only if it can be expressed as a linear combination of positive operator-valued measures.

One of the important applications of our main dilation theorem is the dilation for not necessarily cb-maps with appropriate continuity properties from a commutative von Neumann algebra into $B(\mathcal{H})$. While the ultraweak topology on $B(\mathcal{H})$ for a Hilbert space \mathcal{H} is well-understood, we define the ultraweak topology on $B(X)$ for a Banach space X through tensor products: Let $X \otimes Y$ be the tensor product of the Banach space X and Y . The projective norm on $X \otimes Y$ is defined by:

$$\|u\|_{\wedge} = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

We will use $X \otimes_{\wedge} Y$ to denote the tensor product $X \otimes Y$ endowed with the projective norm $\|\cdot\|_{\wedge}$. Its completion will be denoted by $X \widehat{\otimes} Y$. From [27] Section 2.2, for any Banach spaces X and Y , we have the identification:

$$(X \widehat{\otimes} Y)^* = B(X, Y^*).$$

Thus $B(X, X^{**}) = (X \widehat{\otimes} X^*)^*$. Viewing $X \subseteq X^{**}$, we define the *ultraweak topology* on $B(X)$ to be the weak* topology induced by the predual $X \widehat{\otimes} X^*$. We will also use the term *normal* to denote an ultraweakly continuous linear map.

Theorem 4.5. *If \mathcal{A} is a purely atomic abelian von Neumann algebra acting on a separable Hilbert space, then for every ultraweakly continuous linear map $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$, there exists a Banach space Z , an ultraweakly continuous unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$.

The proof of this theorem uses some special properties of the minimal dilation system for the ϕ induced operator valued measure on the space $(\mathbb{N}, 2^{\mathbb{N}})$. Motivated by some ideas used in the proof of the above theorem, we then obtained a *universal dilation theorem* for all bounded linear mappings between Banach algebras:

Theorem 4.6. *Let \mathcal{A} be a Banach algebra, and let $\phi : \mathcal{A} \rightarrow B(X)$ be a bounded linear operator, where X is a Banach space. Then there exists a Banach space Z , a bounded linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : X \rightarrow Z$ and $S : Z \rightarrow X$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$.

Since this theorem is so general we would expect that there is a also purely algebraic dilation theorem for any linear transformations. This indeed is the case.

Proposition 4.7. *If A is unital algebra, V a vector space, and $\phi : A \rightarrow L(V)$ a linear map, then there exists a vector space W , a unital homomorphism $\pi : A \rightarrow L(W)$, and linear maps $T : V \rightarrow W$, $S : W \rightarrow V$, such that*

$$\phi(\cdot) = S\pi(\cdot)T.$$

This result maybe well-known. However we provide a short proof for interested readers.

Proof. For $a \in A, x \in V$, define $\alpha_{a,x} \in L(A, V)$ by

$$\alpha_{a,x}(\cdot) := \phi(\cdot a)x.$$

Let $W := \text{span}\{\alpha_{a,x} : a \in A, x \in V\} \subset L(A, V)$. Define $\pi : A \rightarrow L(W)$ by $\pi(a)(\alpha_{b,x}) := \alpha_{ab,x}$. It is easy to see that π is a unital homomorphism. For $x \in V$ define $T : V \rightarrow L(A, V)$ by $T_x := \alpha_{I,x} = \phi(\cdot I)x = \phi(\cdot)x$. Define $S : W \rightarrow W$ by setting $S(\alpha_{a,x}) := \phi(a)x$ and extending linearly to W . If $a \in A, x \in V$ are arbitrary, we have $S\pi(a)Tx = S\pi(a)\alpha_{I,x} = S\alpha_{a,x} = \phi(a)x$. Hence $\phi = S\pi T$. \square

We note that the above proposition has been generalized by the second author and F. Szafraniec [21] to the case where A is a unital semigroup.

Theorem 4.6 is a true generalization of our commutative theorem in an important special case, and generalizes some of our results for maps of commutative von Neumann algebras to the case where the von Neumann algebra is non-commutative.

For the case when \mathcal{A} is a von Neumann algebra acting on a separable Hilbert space and ϕ is ultraweakly continuous (i.e., normal) we conjecture that the dilation space Z can be taken to be separable and the dilation homomorphism π is also ultraweakly continuous. While we are not able to confirm this conjecture we have the following result. Here, SOT is the abbreviation of strong operator topology.

Theorem 4.8. *Let K, H be Hilbert spaces, $A \subset B(K)$ be a von Neumann algebra, and $\phi : A \rightarrow B(H)$ be a bounded linear operator which is ultraweakly-SOT continuous on the unit ball B_A of A . Then there exists a Banach space Z , a bounded linear homeomorphism $\pi : A \rightarrow B(Z)$ which is SOT-SOT continuous on B_A , and bounded linear operator $T : H \rightarrow Z$ and $S : Z \rightarrow H$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in A$. If in addition that K, H are separable, then the Banach space Z can be taken to be separable.

These results are apparently new for mappings of von Neumann algebras. They generalize special cases of Stinespring's Dilation Theorem. The standard discrete Hilbert space frame theory is identified with the special case of our theory in which the domain algebra is abelian and purely atomic, the map is completely bounded, and the OVM is purely atomic and completely bounded with rank-1 atoms.

The universal dilation result has connections with Kadison's similarity problem for bounded homomorphisms between von Neumann algebras (see the Remark 4.14). For example, if \mathcal{A} belongs to one of the following classes: nuclear; $\mathcal{A} = B(H)$; \mathcal{A} has no tracial states; \mathcal{A} is commutative; II_1 -factor with Murry and von Neumann's property Γ , then any non completely bounded map $\phi : \mathcal{A} \rightarrow B(H)$ can never have a Hilbertian dilation (i.e. the dilation space Z can never be a Hilbert space) since otherwise $\pi : \mathcal{A} \rightarrow B(Z)$ would be similar to a *-homomorphism and hence completely bounded and so would be ϕ . On the other hand, if there exists a von Neumann algebra \mathcal{A} and a non completely bounded map ϕ from \mathcal{A} to $B(H)$ that has a Hilbert space dilation: $\pi : \mathcal{A} \rightarrow B(Z)$ (i.e., where Z is a Hilbert space), then π will be a counterexample to the Kadison's similarity problem since in this case π is a homomorphism that is not completely bounded and consequently can not be similar to a *-homomorphism.

5. SOME REMARKS AND PROBLEMS

Remark 5.1. It is well known that there is a theory establishing a connection between general bounded linear mappings from the C^* -algebra $C(X)$ of continuous functions on a compact Hausdorff space X into $B(H)$ and operator valued measures on the sigma algebra of Borel subsets of X (c.f. [23]). If A is an abelian C^* -algebra then A can be identified with $C(X)$ for a topological space X and can also be identified with $C(\beta X)$ where βX is the Stone-Cech compactification of X . Then the support σ -algebra for the OVM is the

sigma algebra of Borel subsets of βX which is enormous. However in our generalized (commutative) framing theory \mathcal{A} will always be an abelian von Neumann algebra presented up front as $L^\infty(\Omega, \Sigma, \mu)$, with Ω a topological space and Σ its algebra of Borel sets, and the maps on A into $B(H)$ are normal. In particular, to model the discrete frame and framing theory Ω is a countable index set with the discrete topology (most often \mathbb{N}), so Σ is its power set, and μ is counting measure. So in this setting it is more natural to work directly with this presentation in developing dilation theory rather than passing to $\beta\Omega$, and we took this approach in our investigation.

Remark 5.2. We feel that the connection we make with established discrete frame and framing theory is transparent, and then the OVM dilation theory for the continuous case becomes a natural but nontrivial generalization of the theory for the discrete case that was inspired by framings. After doing this we attempted to apply our techniques to the case where the domain algebra for a map is non-commutative. However, additional hypotheses are needed if dilations of maps are to have strong continuity and structural properties. For a map between C^* -algebras it is well-known that there is a Hilbert space dilation if the map is completely bounded. (If the domain algebra is commutative this statement is an iff.) Even if a map is not cb it has a Banach space dilation. We are interested in the continuity and structural properties a dilation can have. In the discrete abelian case, the dilation of a normal map can be taken to be normal and the dilation space can be taken to be separable, and with suitable hypotheses this type of result can be generalized to the noncommutative setting.

The following is a list of problems we think may be important for the general dilation theory of operator valued measures and bounded linear maps.

It was proven in [18] that if $\{i\}$ is an atom in Σ and E is an operator valued frame on Σ , then the minimal dilation F_α has the property that the rank of $F_\alpha(\{i\})$ is equal to the rank of $E(\{i\})$. This leads to the following problem.

Problem 1. Is it always true that with an appropriate notion of rank function for an operator valued measure, that $r(F_\alpha(B)) = r(E(B))$ for every $B \in \Sigma$? What about if a “rank” definition is defined by: $r(B) = \sup\{\text{rank} E(A) : A \subset B, A \in \Sigma\}$?

Let (Ω, Σ, μ) be a probability space and let $\phi : L^\infty(\mu) \rightarrow B(\mathcal{H})$ be ultraweakly continuous. Then it naturally induces an operator valued probability measure

$$E(B) = \phi(\chi_B), \quad \forall B \in \Sigma.$$

Problem 2. Let $E : (\Omega, \Sigma) \rightarrow B(\mathcal{H})$ be an operator valued measure. Is there an ultraweakly continuous map $\phi : L^\infty(\mu) \rightarrow B(\mathcal{H})$ that induces E on (Ω, Σ) ? If the answer is negative, then determine necessary and sufficient conditions for E to be induced by an ultraweakly continuous map?

As with Stinespring’s dilation theorem, if A and \mathcal{H} in Theorem 4.1 are both separable then the dilated Banach space Z is also separable. However, the Banach algebras we are interested in include von Neumann algebras and these are generally not separable, and the linear maps $\phi : A \rightarrow B(H)$ are often normal. So we pose the following two problems.

Problem 3. Let K, H be separable Hilbert spaces, let $A \subset B(K)$ be a von Neumann algebra, and let $\phi : A \rightarrow B(H)$ be a bounded linear map. When is there a *separable* Banach space Z , a bounded linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$?

Problem 4. Let $A \subset B(K)$ be a von Neumann algebra, and $\phi : A \rightarrow B(H)$ be a normal linear map. When can we dilate ϕ to a *normal* linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$ for some (reflexive) Banach space Z ?

Finally, concerning the Hilbert space dilations and Kadison's Similarity Problem, we are interested in the following questions:

Problem 5. Let $A \subset B(K)$ be a von Neumann algebra, and let $\phi : A \rightarrow B(H)$ be a bounded linear map. We know that ϕ has a Hilbert space dilation if it is completely bounded. Is there a non-completely bounded map that admits a Hilbert space dilation? In particular, if $\phi(A) = A^t$ for any $A \in \oplus_{n=1}^{\infty} M_{n \times n}(\mathbb{C})$, then T is bounded but not completely bounded. What can we say about the dilation of ϕ ? Does it admit a Hilbert space dilation?

Yes. An affirmative answer would yield a negative answer to the similarity problem.

Problem 6. Let $A \subset B(K)$ be a von Neumann algebra, and let $\phi : A \rightarrow B(H)$ be a bounded linear map. "Characterize" those maps that admit Hilbert space dilations, and "Characterize" those maps that admit reflexive Banach space dilations.

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